We must now find such values for t as will make $\frac{6t}{t^2-1}$ a perfect square. These values are 0, 1, 2, 3, $\frac{1}{2}$, $\frac{1}{3}$. For $t=2$, $z=2$,

$$d=\frac{2^6.5^2}{3^2},\ x=\frac{2^8.5}{3^2},\ \text{and}\ y=\frac{2^6.5}{3}.$$

For $t=3$, $z=\frac{3}{2}$, $d=\frac{2^6.5^3}{3^2}$, $x=\frac{2^6.5}{3}$, and $y=\frac{2^8.5^2}{3^2}$.

The values of 0 and 1 for t, give reciprocal limiting values for d, x, and y.

A similar treatment for B leads to the value, $z=2\sqrt{\frac{15t}{t^2-1}}$.
z is rational for $t=0$, 1, 4, $-\frac{1}{4}$, giving the values $z=0$, ∞, 4.
The values of x and y corresponding to $z=4$, are

$$x=\frac{3^2.19.179}{2^3\times5},\ y=\frac{3\times19.179}{5^2},\ \text{diagonal}=\frac{3^2.17.19.179}{2^3\times5}.$$

171. Proposed by PROFESSOR E. B. ESCOTT, Ann Arbor, Mich.

Solve completely:

$$\begin{aligned}2x^2-1&=y,\\2y^2-1&=z,\\2z^2-1&=w,\\2w^2-1&=x.\end{aligned}$$

I. Solution by PROFESSOR L. E. DICKSON, Ph. D., The University of Chicago.

It will be shown that the only integral solution is $x=y=z=w=1$.

If $x=0$ or ±1, then $y^2=1$, $z=1$, $w=1$, so that the fourth equation gives $x=1$.

Next, let $x^2>1$. Then $y>x^2$, $z>y^2>1$, $w>z^2>1$, $x>w^2$.
Hence, $x>x^{16}$, which contradicts $x^2>1$.

Also similarly solved by G. B. M. Zerr, and S. G. Barton.

II. Solution by the PROPOSER.

Let $x=\cos\phi$; then $y=\cos2\phi$, $z=\cos4\phi$, $w=\cos8\phi$, $x=\cos16\phi$.
Since $\cos16\phi=\cos\phi$, we have, $\cos16\phi-\cos\phi=0$, which may be written

$$-2\sin\tfrac{17}{2}\phi.\sin\tfrac{15}{2}\phi=0.$$

If $\sin\frac{17}{2}\phi=0$, $\phi=\frac{2n\pi}{17}$. If $\sin\frac{15}{2}\phi=0$, $\phi=\frac{2n\pi}{15}$.

Taking $\phi=\frac{2n\pi}{15}$, we get seven roots, viz,

$$x_1=\cos\frac{2\pi}{15}=\cos 24^\circ=\tfrac{1}{8}[\sqrt{(30-6\sqrt{5})}+\sqrt{5}+1];$$

$$x_2=\cos\frac{4\pi}{15}=\cos 48^\circ=\tfrac{1}{8}[\sqrt{(30+6\sqrt{5})}-\sqrt{5}+1];$$

$$x_3=\cos\frac{6\pi}{15}=\cos 72^\circ=\tfrac{1}{4}(\sqrt{5}-1);$$

$$x_4=\cos\frac{8\pi}{15}=\cos 96^\circ=-\tfrac{1}{8}[\sqrt{(30-6\sqrt{5})}-\sqrt{5}-1);$$

$$x_5=\cos\frac{10\pi}{15}=\cos 120^\circ=-\tfrac{1}{2};$$

$$x_6=\cos\frac{12\pi}{15}=\cos 144^\circ=-\tfrac{1}{4}(\sqrt{5}+1);$$

$$x_7=\cos\frac{14\pi}{15}=\cos 168^\circ=-\tfrac{1}{8}[\sqrt{(30+6\sqrt{5})}+\sqrt{5}-1).$$

Taking $\phi=\frac{2n\pi}{17}$, we get eight roots. Gauss has shown how the complex roots of $x^{17}-1=0$ may be found by solving a chain of quadratic equations If ω is a complex root of $x^{17}-1=0$, then

$$\omega=\cos\frac{2\pi}{17}+i\sin\frac{2\pi}{17}, \text{ and } \omega^{16}=\cos\frac{2\pi}{17}-i\sin\frac{2\pi}{17},$$

from which we have

$$\cos\frac{2\pi}{17}=\tfrac{1}{2}(\omega+\omega^{16})=x_8;$$

and similarly,

$$\cos\frac{4\pi}{17}=\tfrac{1}{2}(\omega^2+\omega^{15})=x_9,$$

$$\cos\frac{6\pi}{17}=\tfrac{1}{2}(\omega^3+\omega^{14})=x_{10},$$

$$\cos\frac{8\pi}{17}=\tfrac{1}{2}(\omega^4+\omega^{13})=x_{11};$$

$$\cos\frac{10\pi}{17}=\tfrac{1}{2}(\omega^5+\omega^{12})=x_{12};$$

$$\cos\frac{12\pi}{17}=\tfrac{1}{2}(\omega^6+\omega^{11})=x_{13};$$

$$\cos\frac{14\pi}{17}=\tfrac{1}{2}(\omega^7+\omega^{10})=x_{14};$$

$$\cos\frac{16\pi}{17}=\tfrac{1}{2}(\omega^8+\omega^9)=x_{15};$$

We have to solve the chain of equations:

$k^2+k-4=0$. Roots k_1, k_2.
$l^2-k_1l-1=0$. Roots l_1, l_2.
$l^2-k_2l-1=0$. Roots l_3, l_4.
$4x^2-2l_1x+l_3=0$. Roots x_8, x_{11}.
$4x^2-2l_2x+l_4=0$. Roots x_9, x_{15}.
$4x^2-2l_3x+l_2=0$. Roots x_{10}, x_{12}.
$4x^2-2l_4x+l_1=0$. Roots x_{13}, x_{14}.

These fifteen roots together with the root $x=1$ make the sixteen roots. From symmetry, the sets of values for y, z, and w, are the same as for x.

PROBLEMS FOR SOLUTION.

ALGEBRA.

337. Proposed by I. M. CURTISS, Brooklyn, N. Y.

Three regiments move north as follows: B is 20 miles east of A; C is 20 miles south of B, and each marches 20 miles between the hours of 5 a. m. and 3 p. m. A horseman with a message from C starts at 5 a. m. and rides north till he overtakes B, then sets a straight course for the point at which he calculates to overtake A, then sets a straight course for the next point at which he will again overtake B, then rides south to the point where he first overtook B, reaching that point at the same time as C, namely 3 p. m. What uniform rate of travel enabled the messenger to do this?

338. Proposed by R. D. CARMICHAEL, Princeton University.

Prove that $\pi=3+\frac{1}{3}\cdot\frac{1}{1.2}-\frac{1}{5}\cdot\frac{1}{2.3}+\frac{1}{7}\cdot\frac{1}{3.4}-\frac{1}{9}\cdot\frac{1}{4.5}+\ldots$

339. Proposed by E. B. ESCOTT, University of Michigan, Ann Arbor, Mich.

Prove that if $a_1 \not< 2$ and $a_n=a_{n-1}^2-2$, $\frac{1}{a_1}+\frac{1}{a_1a_2}+\frac{1}{a_1a_2a_3}+\ldots$ $=\frac{1}{2}[a_1-\sqrt{(a_1^2-4)}]$.